§4.4 The Magnetic Moment of the Electron

Recall the Dirac equation in the presence of the electromagnetic field:

$$(i\gamma^\mu D_\mu - m)\psi = 0$$

$\rightarrow$ acting on with $(i\gamma^\mu D_\mu + m)$ gives

$$-(\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2)\psi = 0$$

We have

$$\begin{aligned}
&\gamma^\mu\gamma^\nu D_\mu D_\nu \\
&= \frac{1}{2}\left(\{\gamma^\mu,\gamma^\nu\} + [\gamma^\mu,\gamma^\nu]\right) D_\mu D_\nu \\
&= D_\mu D^\mu - i\sigma^{\mu\nu} D_\mu D_\nu
\end{aligned}$$

and

$$\begin{aligned}
i\sigma^{\mu\nu} D_\mu D_\nu &= \left(\frac{i}{2}\right)\sigma^{\mu\nu}[D_\mu, D_\nu] \\
&= \frac{e}{2}\sigma^{\mu\nu} F_{\mu\nu}
\end{aligned}$$

$$\rightarrow \left(D_\mu D^\mu - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} + m^2\right)\psi = 0 \quad (1)$$

Now consider a weak constant magnetic field pointing in the 3-direction:

$$A_0 = 0, \quad A_1 = -\frac{1}{2}Bx^2, \quad A_2 = \frac{1}{2}Bx^1$$

$$\longrightarrow \quad F_{12} = \partial_1 A_2 - \partial_2 A_1$$

then

$$(D_i)^2 = (\partial_i)^2 - ie(\partial_i A_i + A_i \partial_i) + \mathcal{O}(A_i^2)$$

$$= (\partial_i)^2 - 2\frac{ie}{2}B(x^1\partial_2 - x^2\partial_1) + \mathcal{O}(A_i^2)$$

$$= \vec{\nabla}^2 - e\vec{B}\cdot\underbrace{\vec{x}\times\vec{p}}_{=\vec{L}} + \mathcal{O}(A_i^2)$$

where we used

$$\partial_i A_i + A_i\partial_i = (\partial_i A_i) + 2A_i\partial_i = 2A_i\partial_i$$

Next, we write $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ in Dirac basis $\longrightarrow$ in non-relativistic limit $\chi \ll \phi$

recall

$$\sigma^{ij} = \varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

Thus $\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}$ acting on ϕ gives

$$\frac{e}{2}\sigma^3(F_{12} - F_{21}) = \frac{e}{2}2\sigma^3 B = 2e\vec{B}\cdot\vec{S}$$

since $\vec{S} = \frac{\vec{\sigma}}{2}$.

We can write $\phi = e^{-imt}\psi$ where

ψ oscillates much more slowly than e^{-imt}

$\rightarrow$ $(\partial_0^2 + m^2) e^{-imt} \psi \sim e^{-imt} (-2im \frac{\partial}{\partial t} \psi)$

Altogether, we get

$$\left[-2im \frac{\partial}{\partial t} - \vec{\nabla}^2 - e\vec{B} \cdot (\vec{L} + 2\vec{S}) \right] \psi = 0 \quad (2)$$

Dirac eq. tells us that a unit of "spin angular momentum" interacts <u>twice</u> as much as unit of "orbital angular mom."

$\rightarrow$ confirmed experimentally!

Anothe approach: Gordon decomp. (Homework 4):

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p') \left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m} \right] u(p)$$

$\rightarrow$ interaction with electromagnetic field

$$\bar{u}(p')\gamma^\mu u(p) A_\mu(p'-p)$$

contains a magnetic moment component

The anomalous magnetic moment

Experimental magnetic moment is larger than the one calculated by Dirac by a factor 1.00118 ± 0.00003

Let us denote an electron state by

$$|p, s\rangle$$

mom. polarization

→ by Lorentz invariance and current conservation:

$$\langle p', s' | J^\mu(0) | p, s\rangle$$

$$= \bar{u}(p', s')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)\right]u(p,s) \tag{3}$$

where $q := p' - p$

$F_1(q^2)$ and $F_2(q^2)$ are known as form factors.

→ to leading order in q, (3) becomes

$$\bar{u}(p',s')\left\{\frac{(p'+p)^\mu}{2m}F_1(0)+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[F_1(0)+F_2(0)\right]\right\}u(p,s)$$

by Gordon decomposition.

first term: determines electric charge

→ set to $F_1(0)=1$

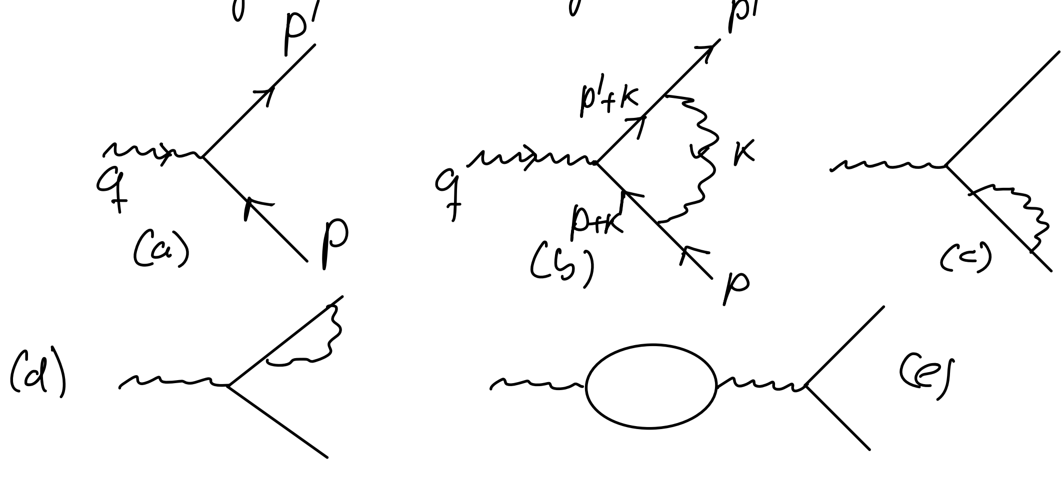
second term: $\frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[1+F_2(0)\right]$

→ magnetic moment of electron is shifted by factor $1+F_2(0)$

<u>Schwinger's calculation</u>

Let us now calculate $F_2(0)$ to order $\alpha=\frac{e^2}{4\pi}$

→ Feynman diagrams:



$\rightarrow$ except for figure (b), all diagrams are proportional to $\bar{u}(p', s')\gamma^\mu u(p, s)$

$\rightarrow$ contribute to $F_1(q^2)$

(b) contributes to $F_2(q^2)$

Write (a) + (b) = $\bar{u}(\gamma^\mu + \Gamma^\mu)u$

$\rightarrow$ applying Feynman rules, find

$$\Gamma^\mu = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2}\left(ie\gamma^\nu \frac{i}{\not{p}' + \not{k} - m} \gamma^\mu \frac{i}{\not{p} + \not{k} - m} ie\gamma_\nu\right)$$

Simplifying, gives

$$\Gamma^\mu = -ie^2 \int \left[\frac{d^4k}{(2\pi)^4}\right]\left(\frac{N^\mu}{D}\right),$$

where

$$N^\mu = \gamma^\nu(\not{p}' + \not{k} + m)\gamma^\mu(\not{p} + \not{k} - m)\gamma_\nu \tag{4}$$

and

$$\frac{1}{D} = \frac{1}{(\not{p}' + k)^2 - m^2} \frac{1}{(p+k)^2 - m^2} \frac{1}{k^2}$$

Using the identity

$$\frac{1}{xyz} = 2\iint_\Delta d\alpha\, d\beta \frac{1}{[z + \alpha(x-z) + \beta(y-z)]^3}$$

where the integration domain is the triangle $0 \leq \beta \leq 1-\alpha$, $0 \leq \alpha \leq 1$ in α-β plane,

gives

$$\frac{1}{D} = 2\int d\alpha\, d\beta \; \frac{1}{\mathcal{D}}$$

with $\mathcal{D} = [k^2 + 2k(\alpha p' + \beta p)]^3$

$$= [\ell^2 - (\alpha+\beta)^2 m^2]^3 + \mathcal{O}(q^2)$$

where we defined $k = \ell - (\alpha p' - \beta p)$

From Gordon decomposition, we know that

$$\bar{u}\,(\gamma^\mu + \Gamma^\mu)\,u$$

$$= \bar{u}\left\{\gamma^\mu[F_1(q^2) + F_2(q^2)] - \frac{1}{2m}(p'+p)^\mu F_2(q^2)\right\}u \quad (5)$$

$\rightarrow$ to extract $F_2(0)$, we can discard any term proportional to γ^μ

rewriting (4) in terms of ℓ, gives

$$N^\mu = \gamma^\nu[\not{\ell} + \not{p}' + m]\gamma^\mu[\not{\ell} + \not{p} + m]\gamma_\nu \quad (6)$$

with $\not{P}'^{\mu} := (1-\alpha)p'^{\mu} - \beta p^{\mu}$

and $P^{\mu} := (1-\beta)p^{\mu} - \alpha p'^{\mu}$

Perform the following steps in (6):

1) the m^2 term: a γ^{μ} term $\to$ discard

2) the m terms: organize by powers of ℓ

$\to$ term linear in ℓ vanishes

$\to$ term independent of ℓ:

$$m(\gamma^{\nu}\not{P}'\gamma^{\mu}\gamma_{\nu} + \gamma^{\nu}\gamma^{\mu}\not{P}\gamma_{\nu})$$

$$= 4m\left[(1-2\alpha)p'^{\mu} + (1-2\beta)p^{\mu}\right]$$

$$\to 4m(1-\alpha-\beta)(p'+p)^{\mu}$$

(used that $\mathcal{D}$ is symmetric in $\alpha \leftrightarrow \beta$)

3) m^0 term: of order ℓ^2

$\to$ by Lorentz invariance replace

$$\int \frac{d^4\ell}{(2\pi)^4} \ell^{\sigma}\ell^{\tau} \to \frac{1}{4}\int \frac{d^4\ell}{(2\pi)^4} \eta^{\sigma\tau}\ell^2$$

$\to$ discard γ^{μ} term and linear terms in ℓ leaving

$$\gamma^{\nu}\not{P}'\gamma^{\mu}\not{P}\gamma_{\nu} = -2\not{P}\gamma^{\mu}\not{P}'$$

$$\longrightarrow -2\left[(1-\beta)\not{p}-\alpha m\right]\gamma^\mu\left[(1-\alpha)\not{p}'-\beta m\right]$$

Putting everything together, we find

$$N^\mu \longrightarrow 2m(p'+p)^\mu(\alpha+\beta)(1-\alpha-\beta)$$

Performing the integral $\int\left[\frac{d^4\ell}{(2\pi)^4}\right]\frac{1}{\mathcal{D}}$, we get

$$\Gamma^\mu = -2ie^2\int d\alpha\, d\beta \left(\frac{-i}{32\pi^2}\right)\frac{1}{(\alpha+\beta)^2 m^2}N^\mu$$

$$= -\frac{e^2}{8\pi^2}\frac{1}{2m}(p'+p)^\mu$$

Finally from (5):

$$F_2(0) = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}$$